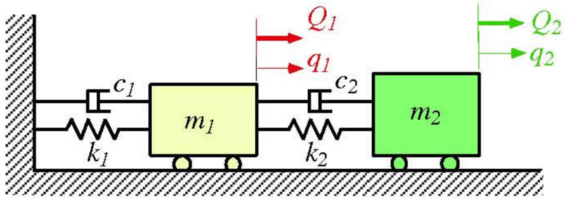
ارتعاشات سیستم‌های چند درجه آزادی با استهلاک دلخواه : مثال

Note Title 3/15/2020

Solution of Nonconservative Eigenvalue Problem as Applied to a 2 dof System with Masses, Dampers, and Springs

A 2 dof mass-damper-spring mechanical system with two masses, dampers, and springs is shown in Figure 1. The generalized coordinates are q_1 and q_2 measured from the equilibrium position, as shown in the figure. The external forces, Q_1 and Q_2 are as shown.



$Q_1 = Q_2 = 0$

Figure 1: A 2 dof mass-damper-spring system for nonconservative eigenvalue problem. The parameters are: $m_1 = 1\,kg$, $m_2 = 2\,kg$, $c_1 = 24\,N \cdot s/m$, $c_2 = 20\,N \cdot s/m$, $k_1 = 3600\,N/m$, and $k_2 = 1600\,N/m$.

The equations of motion of the 2 dof system can be derived and arranged in the standard matrix form as that in equation (1) to yield

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{23}$$

where $\mathbf{q} = [q_1\ q_2]^T$ and $\mathbf{Q} = [Q_1\ Q_2]^T$. Note that the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$ are symmetric and positive definite. Substituting the following parameters into equation (23): $m_1 = 1\,kg$, $m_2 = 2\,kg$, $c_1 = 24\,N \cdot s/m$, $c_2 = 20\,N \cdot s/m$, $k_1 = 3600\,N/m$, and $k_2 = 1600\,N/m$, we have

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 44 & -20 \\ -20 & 20 \end{bmatrix}, \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 5200 & -1600 \\ -1600 & 1600 \end{bmatrix} \tag{24}$$

Equation (23) can be re-arranged in the form of linear system equation as that in equation (2) with $\mathbf{x}(t) = [\mathbf{q}(t)\ \dot{\mathbf{q}}(t)]^T = [q_1(t)\ q_2(t)\ \dot{q}_1(t)\ \dot{q}_2(t)]^T$. The $\mathbf{A}$ matrix from equation (3) becomes

با داشتن ماتریسهای جرم، استهلاک و سختی ماتریس ساخته می شود.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5200 & 1600 & -44 & 20 \\ 800 & -800 & 10 & -10 \end{bmatrix} \tag{25}$$

The eigenvalues of $\mathbf{A}$ and the corresponding left and right eigenvectors are:

$$\lambda_1 = -24.29 + 69.70i, \quad \lambda_2 = -24.29 - 69.70i, \quad \lambda_3 = -2.709 + 22.83i, \quad \lambda_4 = -2.709 - 22.83i \tag{26}$$

$$\mathbf{X} = \begin{bmatrix} -0.00437598 - 0.0125564\,i & -0.00437598 + 0.0125564\,i & -0.000304738 - 0.0142254\,i & -0.000304738 + 0.0142254\,i \\ 0.000170542 + 0.00258329\,i & 0.000170542 - 0.00258329\,i & -0.00483766 - 0.0407712\,i & -0.00483766 + 0.0407712\,i \\ 0.981479 & 0.981479 & 0.325614 + 0.0315797\,i & 0.325614 - 0.0315797\,i \\ -0.184198 - 0.0508636\,i & -0.184198 + 0.0508636\,i & 0.943976 & 0.943976 \end{bmatrix} \tag{27}$$

$$\mathbf{Y} = \begin{bmatrix} 0.69597 + 37.5769\,i & 0.69597 - 37.5769\,i & -1.88893 + 2.15385\,i & -1.88893 - 2.15385\,i \\ 0.13013 - 12.9948\,i & 0.13013 + 12.9948\,i & 0.725584 + 11.5257\,i & 0.725584 - 11.5257\,i \\ 0.481512 + 0.149873\,i & 0.481512 - 0.149873\,i & 0.0858819 + 0.0176721\,i & 0.0858819 - 0.0176721\,i \\ -0.165201 - 0.106162\,i & -0.165201 + 0.106162\,i & 0.503159 + 0.0536663\,i & 0.503159 - 0.0536663\,i \end{bmatrix} \tag{28}$$

Equations (4) and (5) can be verified with $\mathbf{X}$ and $\mathbf{Y}$ in equations (27) and (28).

Response of $\mathbf{x}(t)$ to initial conditions: Consider system with no excitation, i.e., $\mathbf{Q} = \mathbf{0}$. The initial conditions of this system are given as follows:

$$\mathbf{q}(0) = \begin{bmatrix} q_0 \\ 0 \end{bmatrix} \qquad \dot{\mathbf{q}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{29}$$

From equation (11), the response of the system in equation (2) to initial conditions is

$$\mathbf{x}(t) = \Phi(t)\,\mathbf{x}(0) = \mathbf{X}\, e^{\mathbf{\Lambda} t}\, \mathbf{Y}^T\, \mathbf{x}(0) \tag{30}$$

where $\mathbf{X}$ is given in equation (27) and $\mathbf{Y}$ in equation (28), and

$$e^{\mathbf{\Lambda} t} = \begin{bmatrix} e^{(-24.2909+69.7001\,i)\,t} & 0 & 0 & 0 \\ 0 & e^{(-24.2909-69.7001\,i)\,t} & 0 & 0 \\ 0 & 0 & e^{(-2.70905+22.8316\,i)\,t} & 0 \\ 0 & 0 & 0 & e^{(-2.70905-22.8316\,i)\,t} \end{bmatrix} \tag{31}$$

Substituting equations (27), (28), (31), and (29) into equation (30), we obtain

$$\begin{aligned} \mathbf{x}(t) \quad = \quad q_0 \Bigg[& (0.468785 + 0.173175\,i)\; e^{(-24.2909-69.7001\,i)\,t} + (0.468785 - 0.173175\,i)\; e^{(-24.2909+69.7001\,i)\,t} + \\ & (-0.0969533 - 0.00820635\,i)\; e^{(-24.2909-69.7001\,i)\,t} + (-0.0969533 + 0.00820635\,i)\; e^{(-24.2909+69.7001\,i)\,t} + \\ & (0.68308 - 36.881\,i)\; e^{(-24.2909-69.7001\,i)\,t} + (0.68308 + 36.881\,i)\; e^{(-24.2909+69.7001\,i)\,t} + \\ & (1.7831 + 6.957\,i)\; e^{(-24.2909-69.7001\,i)\,t} + (1.7831 - 6.957\,i)\; e^{(-24.2909+69.7001\,i)\,t} + \\ & (0.0312151 - 0.0262144\,i)\; e^{(-2.70905-22.8316\,i)\,t} + (0.0312151 + 0.0262144\,i)\; e^{(-2.70905+22.8316\,i)\,t}\} \\ & (0.0969533 - 0.0665943\,i)\; e^{(-2.70905-22.8316\,i)\,t} + (0.0969533 + 0.0665943\,i)\; e^{(-2.70905+22.8316\,i)\,t}\} \\ & (-0.68308 - 0.641673\,i)\; e^{(-2.70905-22.8316\,i)\,t} + (-0.68308 + 0.641673\,i)\; e^{(-2.70905+22.8316\,i)\,t}\} \\ & (-1.7831 - 2.03319\,i)\; e^{(-2.70905-22.8316\,i)\,t} + (-1.7831 + 2.03319\,i)\; e^{(-2.70905+22.8316\,i)\,t} \Bigg] \end{aligned}$$

After simplification, we obtain

$$\begin{aligned} \mathbf{x}(t) \quad = \quad & q_0\, e^{-2.70905\,t} \begin{bmatrix} 0.0624302\, \cos(22.8316\,t) - 0.0524289\, \sin(22.8316\,t) \\ 0.193907\, \cos(22.8316\,t) - 0.133189\, \sin(22.8316\,t) \\ -1.36616\, \cos(22.8316\,t) - 1.28335\, \sin(22.8316\,t) \\ -3.56621\, \cos(22.8316\,t) - 4.06637\, \sin(22.8316\,t) \end{bmatrix} + \\ & q_0\, e^{-24.2909\,t} \begin{bmatrix} 0.93757\, \cos(69.7001\,t) + 0.34635\, \sin(69.7001\,t) \\ -0.193907\, \cos(69.7001\,t) - 0.0164127\, \sin(69.7001\,t) \\ 1.36616\, \cos(69.7001\,t) - 73.7619\, \sin(69.7001\,t) \\ 3.56621\, \cos(69.7001\,t) + 13.914\, \sin(69.7001\,t) \end{bmatrix} \end{aligned} \tag{32}$$

The results of $\mathbf{x}(t)$ in equation (32) represent the sum of two exponentially decaying curves for $\mathbf{x}(t) = [q_1(t)\; q_2(t)\; \dot{q}_1(t)\; \dot{q}_2(t)]^T$. The results can be plotted and are shown in Figures 2 and 3 for the displacements and velocities, respectively. It can be seen that q_1 and $\dot{q}_1$ settles down more quickly than q_2 and $\dot{q}_2$.

Comparison with the Undamped System: If the damping matrix is set to zero, the conservative system will have the following solution of eigenvalue problem.

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 5200 & -1600 \\ -1600 & 1600 \end{bmatrix} \tag{33}$$

where the units for mass is kg and for stiffness $N \cdot s/m$. The dynamical matrix is

$$\mathbf{D} = \begin{bmatrix} \frac{1}{3600} & \frac{1}{1800} \\ \frac{1}{3600} & \frac{13}{7200} \end{bmatrix} \tag{34}$$

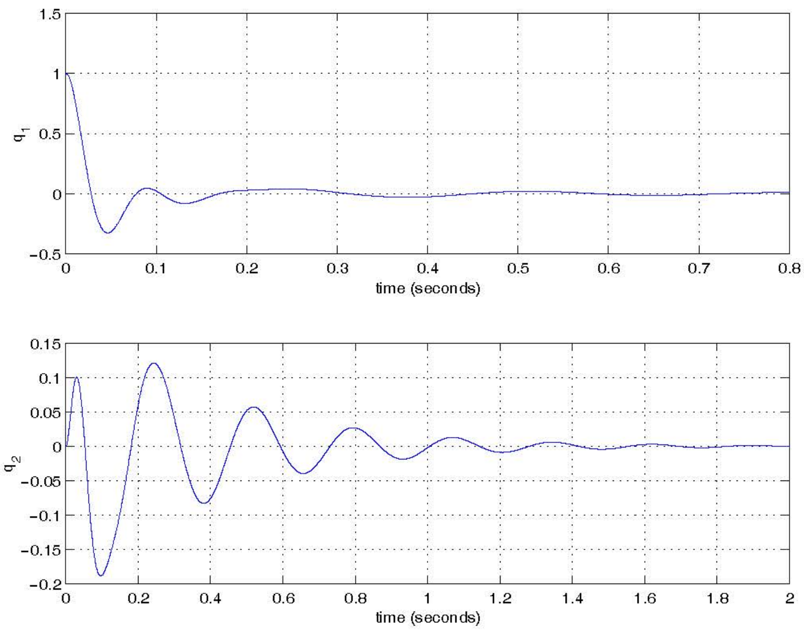


Figure 2: The displacements of q_1 and q_2 for the 2 dof system.

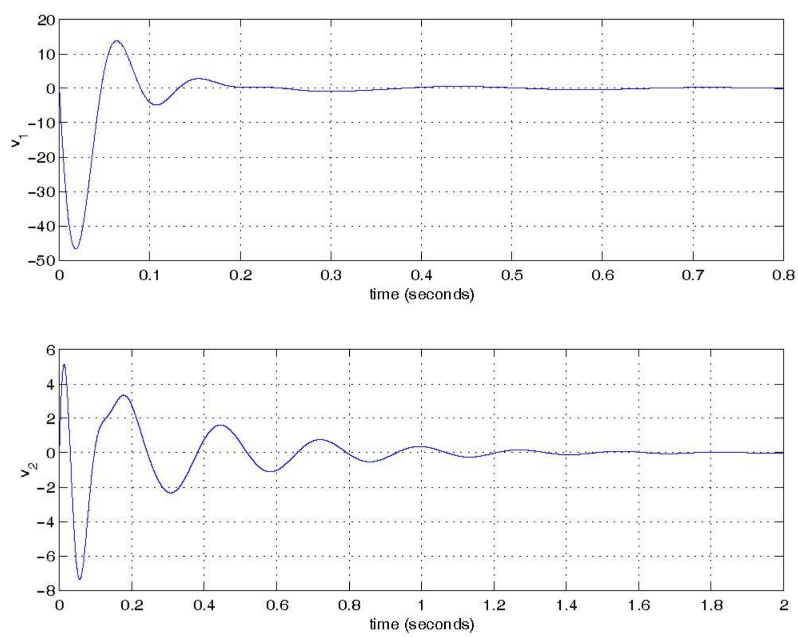


Figure 3: The velocities of $\dot{q}_1$ and $\dot{q}_2$ for the 2 dof system.

The eigenvalues of the dynamical matrix $\mathbf{D}$ are

$$\lambda_1 = 0.00190065, \qquad \lambda_2 = 0.000182686 \tag{35}$$

with corresponding eigenvectors of

$$\mathbf{u}_1 = \begin{bmatrix} -0.323877 \\ -0.946099 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} -0.985666 \\ 0.168711 \end{bmatrix} \tag{36}$$

The natural frequencies are

$$\omega_1 = 22.9377\, rad/sec, \qquad \omega_2 = 73.9856\, rad/sec \tag{37}$$

Comparing with the response in equation (32), we find that the damped frequencies of vibration are 22.8316 rad/sec and 69.7001 rad/sec which are smaller than the natural frequencies in equation (37), as expected for the damped system.