Compute free vibration solution of the following three degree of freedom system

$$\begin{pmatrix} 4. & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{pmatrix} + \begin{pmatrix} 10. & -5 & 0 \\ -5 & 10 & -5 \\ 0 & -5 & 15 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Initial conditions: $\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\begin{pmatrix} \dot{y}_1(0) \\ \dot{y}_2(0) \\ \dot{y}_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

The eigenvalues and eigenvectors of $k - \lambda m$ are as follows.

$\lambda_1 = 0.941275$; $\lambda_2 = 3.0563$; $\lambda_3 = 4.75242$

$$\boldsymbol{Y}^{(1)} = \begin{pmatrix} -0.295505 \\ -0.368488 \\ -0.163993 \end{pmatrix}; \quad \boldsymbol{Y}^{(2)} = \begin{pmatrix} -0.368488 \\ 0.163993 \\ 0.295505 \end{pmatrix}; \quad \boldsymbol{Y}^{(3)} = \begin{pmatrix} 0.163993 \\ -0.295505 \\ 0.368488 \end{pmatrix}$$

$\omega_1 = \sqrt{\lambda_1} = 0.970194$; $\omega_2 = \sqrt{\lambda_2} = 1.74823$; $\omega_3 = \sqrt{\lambda_3} = 2.18001$

The free vibration solution is therefore as follows.

$$y(t) = (A_1 \cos \omega_1 t + A_2 \sin \omega_1 t)\boldsymbol{Y}^{(1)} + (A_3 \cos \omega_2 t + A_4 \sin \omega_2 t)\boldsymbol{Y}^{(2)} + (A_5 \cos \omega_3 t + A_6 \sin \omega_3 t)\boldsymbol{Y}^{(3)}$$

$$\begin{aligned} y_1(\mathrm{t}) = {} & -0.295505\,(\cos(0.970194\,t)\,A_1 + \sin(0.970194\,t)\,A_2) - \\ & 0.368488\,(\cos(1.74823\,t)\,A_3 + \sin(1.74823\,t)\,A_4) + 0.163993\,(\cos(2.18001\,t)\,A_5 + \sin(2.18001\,t)\,A_6) \\ y_2(\mathrm{t}) = {} & -0.368488\,(\cos(0.970194\,t)\,A_1 + \sin(0.970194\,t)\,A_2) + 0.163993\,(\cos(1.74823\,t)\,A_3 + \sin(1.74823\,t)\,A_4) - \\ & 0.295505\,(\cos(2.18001\,t)\,A_5 + \sin(2.18001\,t)\,A_6) \\ y_3(\mathrm{t}) = {} & -0.163993\,(\cos(0.970194\,t)\,A_1 + \sin(0.970194\,t)\,A_2) + 0.295505\,(\cos(1.74823\,t)\,A_3 + \sin(1.74823\,t)\,A_4) + \\ & 0.368488\,(\cos(2.18001\,t)\,A_5 + \sin(2.18001\,t)\,A_6) \end{aligned}$$

Differentiating with respect to t we get

$$\begin{aligned} \dot{y}_1(\mathrm{t}) = {} & -0.295505\,(0.970194\cos(0.970194\,t)\,A_2 - 0.970194\sin(0.970194\,t)\,A_1) - \\ & 0.368488\,(1.74823\cos(1.74823\,t)\,A_4 - 1.74823\sin(1.74823\,t)\,A_3) + \\ & 0.163993\,(2.18001\cos(2.18001\,t)\,A_6 - 2.18001\sin(2.18001\,t)\,A_5) \\ \dot{y}_2(\mathrm{t}) = {} & -0.368488\,(0.970194\cos(0.970194\,t)\,A_2 - 0.970194\sin(0.970194\,t)\,A_1) + \\ & 0.163993\,(1.74823\cos(1.74823\,t)\,A_4 - 1.74823\sin(1.74823\,t)\,A_3) - \\ & 0.295505\,(2.18001\cos(2.18001\,t)\,A_6 - 2.18001\sin(2.18001\,t)\,A_5) \\ \dot{y}_3(\mathrm{t}) = {} & -0.163993\,(0.970194\cos(0.970194\,t)\,A_2 - 0.970194\sin(0.970194\,t)\,A_1) + \\ & 0.295505\,(1.74823\cos(1.74823\,t)\,A_4 - 1.74823\sin(1.74823\,t)\,A_3) + \\ & 0.368488\,(2.18001\cos(2.18001\,t)\,A_6 - 2.18001\sin(2.18001\,t)\,A_5) \end{aligned}$$

Using the six initial conditions we get the following system of equations

$$y_1(0) = 0 = -0.295505\,A_1 - 0.368488\,A_3 + 0.163993\,A_5$$
$$y_2(0) = 1 = -0.368488\,A_1 + 0.163993\,A_3 - 0.295505\,A_5$$
$$y_3(0) = 0 = -0.163993\,A_1 + 0.295505\,A_3 + 0.368488\,A_5$$
$$\dot{y}_1(0) = 1 = -0.286697\,A_2 - 0.644201\,A_4 + 0.357505\,A_6$$
$$\dot{y}_2(0) = 0 = -0.357505\,A_2 + 0.286697\,A_4 - 0.644201\,A_6$$
$$\dot{y}_3(0) = 1 = -0.159105\,A_2 + 0.516609\,A_4 + 0.803306\,A_6$$

Solving these equations simultaneously we get the following values for these constants.

$$\{A_1 \to -1.47395,\ A_2 \to -1.89446,\ A_3 \to 0.655971,\ A_4 \to -0.166989,\ A_5 \to -1.18202,\ A_6 \to 0.977027\}$$

Substituting these values the displacement time histories are as follows.

$$y_1(\text{t}) = -0.295505\,(-1.47395\cos(0.970194\,t) - 1.89446\sin(0.970194\,t)) - 0.368488\,(0.655971\cos(1.74823\,t) - 0.166989\sin(1.74823\,t)) + 0.163993\,(0.977027\sin(2.18001\,t) - 1.18202\cos(2.18001\,t))$$
$$y_2(\text{t}) = -0.368488\,(-1.47395\cos(0.970194\,t) - 1.89446\sin(0.970194\,t)) + 0.163993\,(0.655971\cos(1.74823\,t) - 0.166989\sin(1.74823\,t)) - 0.295505\,(0.977027\sin(2.18001\,t) - 1.18202\cos(2.18001\,t))$$
$$y_3(\text{t}) = -0.163993\,(-1.47395\cos(0.970194\,t) - 1.89446\sin(0.970194\,t)) + 0.295505\,(0.655971\cos(1.74823\,t) - 0.166989\sin(1.74823\,t)) + 0.368488\,(0.977027\sin(2.18001\,t) - 1.18202\cos(2.18001\,t))$$