CFD I - جلسهٔ مجازی سوم :

Implicit Methods

روش‌های ضمنی برای حل معادلات سهمی :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \xrightarrow{FTCS} u_i^{n+1} = r u_{i+1}^n + (1-2r) u_i^n + r u_{i-1}^n$$

که $r = \frac{k}{h^2}$

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} + O(\Delta t, \Delta x^2)$$

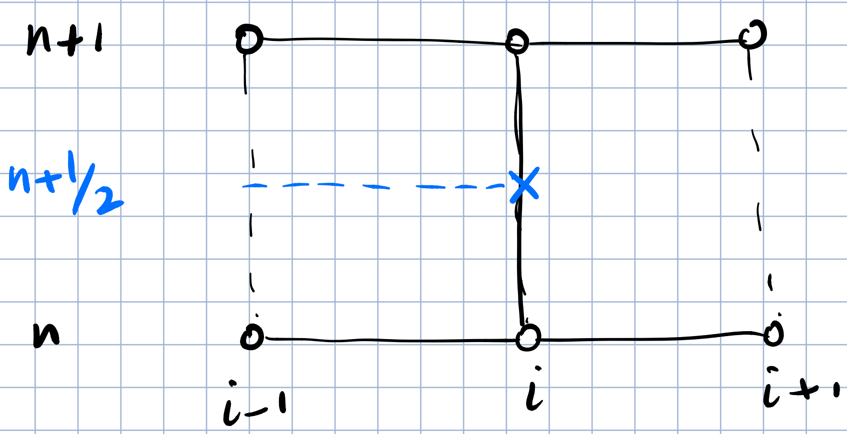
$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{k} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}$$

$$\Rightarrow -r u_{i-1}^{n+1} + (1-2r) u_i^{n+1} - r u_{i+1}^{n+1} = u_i^n$$

$i = 1, 2, \dots, I_{max}$

$$\begin{bmatrix} 1+2r & -r & 0 & \cdots & \cdots & 0 \\ -r & 1+2r & -r & 0 & \cdots & 0 \\ 0 & -r & 1+2 & -r & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & & & & \\ 0 & \cdots & & & -r & 1+2r \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{I_{max}}^{n+1} \end{bmatrix} = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_{I_{max}}^n \end{bmatrix}$$

Crank - Nicolson Method : روش کرنک - نیکلسون



$$\frac{u_i^{n+1} - u_i^{n}}{\Delta t} = \frac{u_{i+1}^{n+1/2} - 2u_i^{n+1/2} + u_{i-1}^{n+1/2}}{\Delta x^2} + O(\Delta t^2, \Delta x^2)$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^{n}}{\Delta t} = \frac{1}{(\Delta x)^2}\left[\left(\frac{u_{i+1}^{n+1} + u_{i+1}^{n}}{2}\right) - 2\left(\frac{u_i^{n+1} + u_i^{n}}{2}\right) + \left(\frac{u_{i-1}^{n+1} + u_{i-1}^{n}}{2}\right)\right]$$

$$\Rightarrow r u_{i-1}^{n+1} - 2(1+r) u_i^{n+1} + r u_{i+1}^{n+1} =$$

$$-r u_{i-1}^{n} + 2(-1+r) u_i^{n} + r u_{i+1}^{n}$$

Combined Method : روش ترکیبی

$$\frac{u_i^{n+1} - u_i^{n}}{\Delta t} = \frac{\frac{1}{2}\delta_x^2 u_i^{n+1} + \frac{1}{2}\delta_x^2 u_i^{n}}{\Delta x^2}$$

$$\frac{u_i^{n+1} - u_i^{n}}{\Delta t} = \frac{\theta\, \delta_x^2 u_i^{n+1} + (1-\theta)\, \delta_x^2 u_i^{n}}{\Delta x^2}$$

$\theta = 0 \Rightarrow$ FTCS (explicit)

$\theta = 1 \Rightarrow$ BTCS (Implicit)

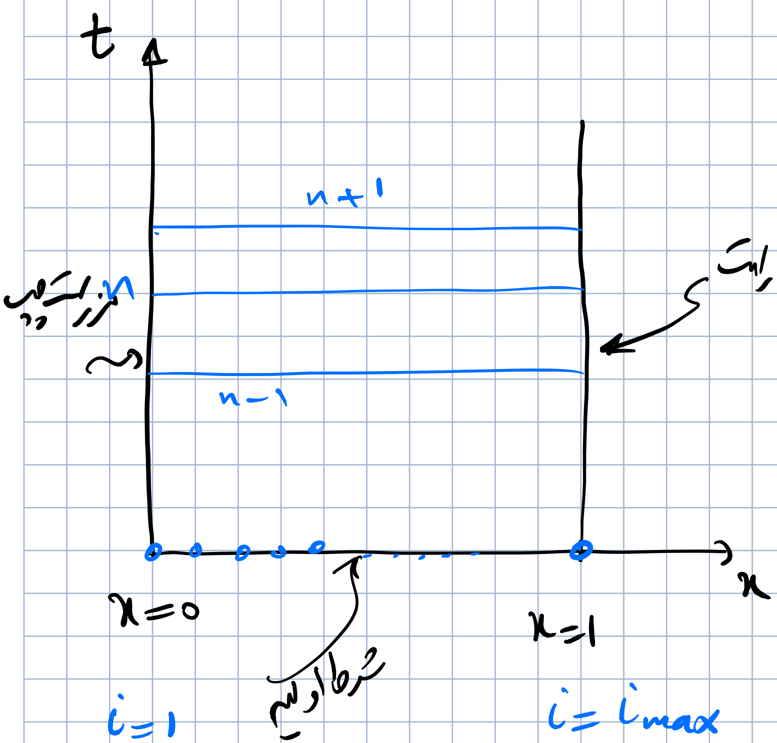
$\theta = 1/2 \Rightarrow$ کرنک-نیکلسون $\Rightarrow T.E. \sim O(\Delta t^2, \Delta x^4)$

اگر $\theta \neq 1/2 \Rightarrow T.E. \sim O(\Delta t, \Delta x^2)$

انواع شرایط مرزی و نحوهٔ اعمال آنها:

شرط مرزی دیریکله:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$



$$u(0,t) = u_L$$

$$u(1,t) = u_R$$

$$u(x,0) = f(x)$$

$$A_i u_{i-1}^{n+1} + B_i u_i^{n+1} + C_i u_{i+1}^{n+1} = R_i^n$$

$$i=1 \Rightarrow u_1^{n+1} = u_L$$

$$i=2 : \Rightarrow A_2 \cancel{u_1^{n+1}} + B_2 u_2^{n+1} + C_2 u_3^{n+1} = R_2^n$$

u_L

$$\Rightarrow \quad B_2 u_2^{n+1} + C_2 u_3^{n+1} = R_2^n - A_2 u_L$$

$$i = i_{max-1} \Rightarrow A_{i_{max-1}} u_{i_{max-2}}^{n+1} + B_{i_{max-1}} u_{i_{max-1}}^{n+1} = R_{i_{max-1}} - C_{i_{max-1}} u_R$$

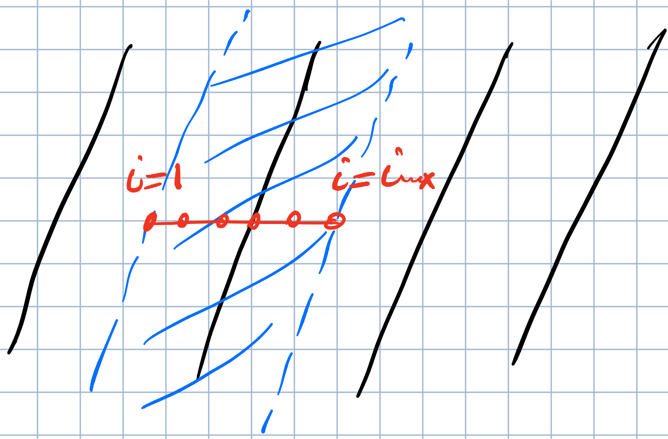
$$i = i_{max} \Rightarrow u_{i_{max}} = u_R$$

$$\begin{bmatrix} B_2 & C_2 & 0 & \cdots & \cdots & 0 \\ A_3 & B_3 & C_3 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & O & & & & O \\ & & A_{i_{max-2}} & B_{i_{max-2}} & C_{i_{max-2}} & \\ 0 & \cdots & 0 & & A_{i_{max-1}} & B_{i_{max-1}} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ \vdots \\ u_{i_{max-2}} \\ u_{i_{max-1}} \end{bmatrix}^{n+1} = \begin{bmatrix} R_2 - A_2 u_L \\ R_3 \\ \vdots \\ \vdots \\ R_{i_{max-2}} \\ R_{i_{max-1}} - C_{i_{max-1}} u_R \end{bmatrix}$$

: For Two

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ A_2 & B_2 & C_2 & \cdots & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & O & & & O & \\ 0 & \cdots & & A_{i_{max-1}} & B_{i_{max-1}} & C_{i_{max-1}} \\ & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{i_{max-1}}^{n+1} \\ u_{i_{max}}^{n+1} \end{bmatrix} = \begin{bmatrix} u_L \\ R_2 \\ \vdots \\ \vdots \\ R_{i_{max-1}} \\ u_R \end{bmatrix}$$

Periodic B.C. شرط مرزی پریودیک :



$$A_i u_{i-1}^{n+1} + B_i u_i^{n+1} + C_i u_{i+1}^{n+1} = R_i$$

$$i=1 \Rightarrow \quad A_1 \underbrace{u_0^{n+1}}_{u_M} + B_1 u_1 + C_1 u_2 = R_1$$

$$\Rightarrow \quad A_1 u_M^{n+1} + B_1 u_1^{n+1} + C_1 u_2^{n+1} = R_1$$

$$i=M \Rightarrow \quad A_M u_{M-1} + B_M u_M + C_M \underbrace{u_{M+1}}_{u_1} = R_M$$

$$\Rightarrow \quad A_M u_{M-1} + B_M u_M + C_M u_1 = R_M$$

$$
\begin{bmatrix}
B_1 & C_1 & 0 & \cdots & \cdots & 0 & A_1 \\
A_2 & B_2 & C_2 & 0 & \cdots & \cdots & 0 \\
 & \ddots & \ddots & \ddots & & 0 & \\
 & 0 & & \ddots & \ddots & \ddots & \\
 & & & & A_{M-1} & B_{M-1} & C_{M-1} \\
C_M & 0 & \cdots & \cdots & & A_M & B_M
\end{bmatrix}
\begin{bmatrix}
u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{M-1} \\ u_M
\end{bmatrix}
=
\begin{bmatrix}
R_1 \\ R_2 \\ \vdots \\ \vdots \\ \vdots \\ R_M
\end{bmatrix}
$$

شرط مرزی مشتق (شرط نیومن) :

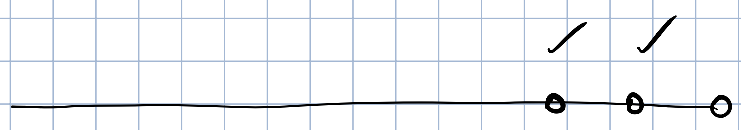
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$
\begin{cases}
u(a,t) = u_a(t) \\
\dfrac{\partial u}{\partial x}(b,t) = u_{x_b}(t)
\end{cases}
$$

الف - روش یکسویه در مرز (one - sided)

ب - روش مرز مجازی (False - Boundary)

روش یکسویه در مرز :



i_{max}

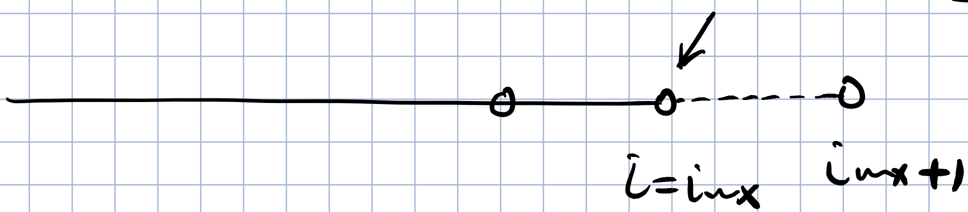
$$\underbrace{\left.\frac{\partial u}{\partial x}\right|_{i=i_{max}}}_{u_{x_b}} = \frac{3u_{i_{max}} - 4u_{i_{max}-1} + u_{i_{max}-2}}{2\Delta x} + O(\Delta x^2)$$

$$\Rightarrow u_{imx} = \frac{2}{3} u_{xb} \Delta x + \frac{1}{3}\left(4u_{imx-1} - u_{imx-2}\right)$$

$$i = imx-1 \Rightarrow A_{imx-1} u^{n+1}_{imx-2} + B_{imx-1} u^{n+1}_{imx-1} + C_{imx-1} u^{n+1}_{imx} = R_{imx-1}$$

$$\Rightarrow \underbrace{\left(A_{imx-1} - \frac{C_{imx-1}}{3}\right)}_{\text{نو } A_{imx-1}} u^{n+1}_{imx-2} + \underbrace{\left(B_{imx-1} + \frac{4}{3} C_{imx-1}\right)}_{B_{imx-1}} u^{n+1}_{imx-1} = R_{imx-1} - \frac{2}{3} C_{imx-1} u_{xb}(\Delta x)$$

روش مرز مجازی:



$$\left.\frac{\partial u}{\partial x}\right|_{i=imx} = \frac{u_{imx+1} - u_{imx-1}}{2\Delta x} = u_{xb}$$

$$\Rightarrow u_{imx+1} = 2u_{xb}\Delta x + u_{imx-1}$$

$$i = imx \Rightarrow A_{imx} u^{n+1}_{imx-1} + B_{imx} u^{n+1}_{imx} + C_{imx} u^{n+1}_{imx+1} = R_{imx}$$

$$\Rightarrow \underbrace{(A_{imx}+C_{imx})}_{\text{نو } A_{imx}} u^{n+1}_{imx-1} + \underbrace{B_{imx}}_{\text{نوین } B_{imx}} u^{n+1}_{imx} = R_{imx} - 2u_{xb} C_{imx} \Delta x$$

✓ الگوریتم توماس

✓ " توماس دیریکله

✓ " توماس کلوی