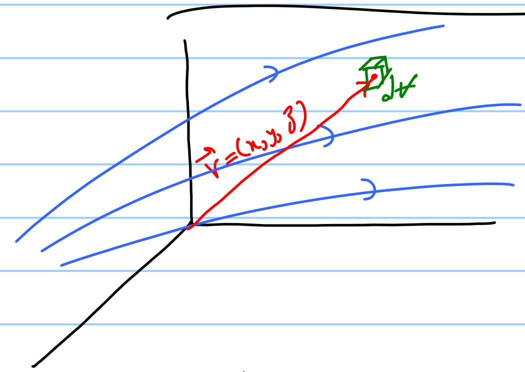
Note Title 4/26/2020

سیالات ۱ - یکشنبه ۹۹/۲/۷

فرم دیفرانسیلی معادله مومنتم خطی :

$$d\vec{F} = dm\,\frac{d\vec{V}}{dt}$$

$$\vec{V} = \vec{V}(x, y, z, t)$$

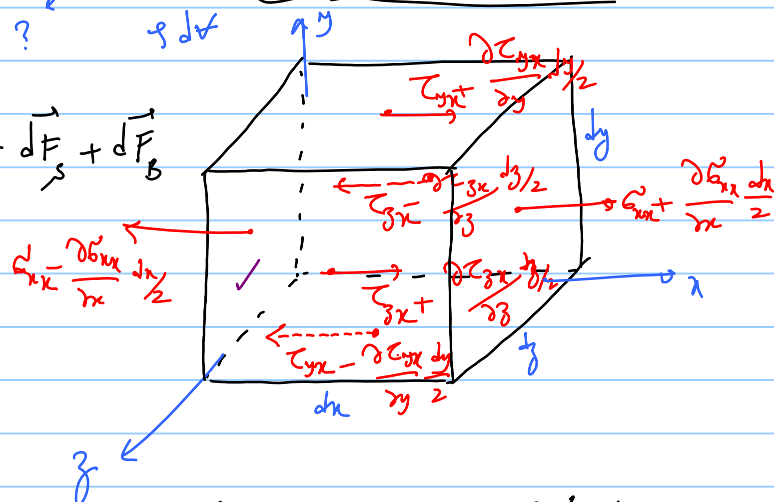


$$\frac{d\vec{V}}{dt} = \vec{a} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

$$\Rightarrow d\vec{F} = dm\left(u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}\right) \leftarrow$$

? $\quad \rho\, d\forall$

$$d\vec{F} = d\vec{F}_S + d\vec{F}_B$$



$$\Rightarrow dF_{S_x} = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dy\,dz - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dy\,dz$$

$$+\left(\tau_{yx} + \frac{\partial \tau}{\partial y}\frac{dy}{2}\right)dx\,dz - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dx\,dz$$

$$+\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy$$

$$\Rightarrow dF_{S_x} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dx\,dy\,dz$$

$$dF_{B_x} = dm\,g_x = (\rho\, d\forall)\,g_x = \rho\, g_x\, dx\,dy\,dz$$

$$\Rightarrow dF_x = F_{Sx} + F_{Bx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x\right) dx\,dy\,dz$$

به صورت مشابه برای راستای y و z خواهیم داشت:

$$\Rightarrow dF_y = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y\right) dx\,dy\,dz$$

$$dF_z = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z\right) dx\,dy\,dz$$

با جایگذاری در معادله مومنتم:

$$\text{x-mom:}\quad \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \underbrace{\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)}_{a_x = \frac{Du}{Dt}}$$

$$\text{y-mom:}\quad \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \quad Dv/Dt$$

$$\text{z-mom:}\quad \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \quad Dw/Dt$$

فرم دیفرانسیلی معادله مومنتم:

با فرض سیال نیوتنی میدانیم که تنش برشی از رابطه زیر بدست می‌آید:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

همچنین رابطه تنش نرمال نیز برابر است با:

$$\sigma_{xx} = -p - \frac{2}{3}\mu \vec{\nabla}\cdot\vec{v} + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p - \frac{2}{3}\mu \vec{\nabla}\cdot\vec{v} + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p - \frac{2}{3}\mu \vec{\nabla}\cdot\vec{v} + 2\mu\frac{\partial w}{\partial z}$$

با جایگذاری تنش برشی و تنش نرمال در معادله فرم دیفرانسیلی معادله مومنتم، معادلات ناویر-استوکس نتیجه خواهد شد:

معادلات ناویر-استوکس :

x-mom: $$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

y-mom: $$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

z-mom: $$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)\right]$$

معادلات ناویر استوکس برای جریان تراکم ناپذیر :

پیوستگی $\Rightarrow \vec{\nabla}\cdot\vec{V} = 0$ ✓

x-mom: $$\Rightarrow \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\nabla^2 u} + \rho g_x$$

$$\Rightarrow \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \rho g_x$$ ✓

y-mom: $$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \rho g_y$$ ✓

z-mom: $$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu\nabla^2 w + \rho g_z$$ ✓

به فرم برداری :

$$\Rightarrow \rho \frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \mu\nabla^2\vec{V} + \rho\vec{g}$$

مجهولات مسئله برای جریان تراکم ناپذیر ⇐ P, u, v, w

فرم معادلات ناویر استوکس برای جریان تراکم ناپذیر در مختصات استوانه‌ای :

$$\rho\underbrace{\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)}_{\vec{a}_r} = -\frac{\partial p}{\partial r} + \rho g_r$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\left(\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_z}{\partial r\partial z}\right)\right]$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\rho g_\theta$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\left(\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)\right]$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$